Bit Error Rate of Coherent *M*-ary PSK

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The bit error rate (BER) for the coherent detection of M-ary PSK signals with Gray code bit mapping is considered. A closed-form expression for the exact BER of 8-ary PSK is presented. Tight upper and lower bounds on BER are also obtained for M-ary PSK with larger M.

I. Introduction

The symbol error rate (SER) for the coherent detection of M-ary PSK signals has been studied very extensively. However, except for the binary and quarternary PSK cases, there has not been much effort in finding the bit error rate (BER), which is a more useful performance measure in many cases. After a brief review of the background, we will derive expressions for the BER of M-ary PSK systems with Gray code bit mapping over an additive white Gaussian noise (AWGN) channel.

II. Preliminaries

Let X be the channel input symbol that takes its value from an M-ary alphabet $\{0, 1, \cdots, M-1\}$. In this study, the channel input alphabet size M is restricted to 2^k so that an M-ary symbol can be represented by k bits. When X is m for a given symbol time, the transmitter sends a sinusoidal signal with phase $2ma_k$, where $a_k = \pi/M$. Let E_b be the received signal energy per information bit and N_0 be the one-sided noise power spectral density of the AWGN. When the transmitted symbol is m, then, with ideal coherent detection, the probability density function of the received signal vector $\mathbf{y} = (y_c, y_s)$ is a two-dimensional Gaussian function, given by

$$p_{\mathbf{Y}}(\mathbf{y}) = f\left\{y_c - d_k \cos\left(2ma_k\right)\right\} \cdot f\left\{y_s - d_k \sin\left(2ma_k\right)\right\}$$

where $d_k = \sqrt{2kE_b/N_0}$ and $f(t) = \exp(-t^2/2)/\sqrt{2\pi}$. For the 8-ary case, such a symbol-to-signal space mapping is shown in Fig. 1. The optimum decision boundaries, which minimize the decision error probability (i.e., SER) with equiprobable input symbols, are also shown with dashed lines. The decision region for $X = m$ will be denoted by R_m.

When BER is the performance measure, the bit assignment for each M-ary symbol is important. The best bits-to-symbol mapping for M-ary PSK signals is the so-called Gray code mapping, with which the k bits corresponding to adjacent symbols differ in only one position. Such a mapping is shown in Table 1 for $k = 2, 3, 4$, and 5 (or $M = 4, 8, 16$, and 32). With Gray code bit mapping, the BER is independent of the transmitted symbols. Hence, without loss of generality, we will assume that 0 is the transmitted symbol. Notice that the decision regions that are optimum for SER also minimize the BER.

Let A_m be the probability that the received signal vector $\mathbf{y}$ falls into R_m when $X = 0$, i.e., $A_m = \Pr\left\{\mathbf{Y} \in R_m \mid X = 0\right\}$. Notice that, for any k, the following relation holds:

$$A_0 > A_1 = A_{M-1} > A_2 = A_{M-2} > \cdots > A_{M/2-1} = A_{M/2+1} > A_{M/2} \tag{1}$$

and these values can be found using one of the following equations:

$$A_m = \int_0^\infty f(z) \cdot \Big\{ Q\big[d_k + z \cdot \tan\big((M/2 - 2m - 1)\, a_k\big)\big]$$

$$- Q\big[d_k + z \cdot \tan\big((M/2 - 2m + 1) a_k\big)\big] \Big\} \cdot dz,$$

$$m = 1, 2, \cdots, M/2 - 1 \tag{2}$$

or

$$A_m = \begin{cases} \displaystyle\int_0^\infty f\big(z - d_k\big) \cdot \Big\{ Q\big[z \cdot \tan\big((2m - 1)\, a_k\big)\big] - Q\big[z \cdot \tan\big((2m + 1)\ a_k\big)\big]\Big\} \cdot dz, \\ \qquad m = 0, 1, \cdots, M/4 - 1 \\ \displaystyle\int_0^\infty f\big(z + d_k\big) \cdot \Big\{ Q\big[z \cdot \tan\big((M - 2m - 1)\, a_k\big)\big] - Q\big[z \cdot \tan\big((M - 2m + 1)\ a_k\big)\big]\Big\} \cdot dz, \\ \qquad m = M/4 + 1, \cdots, M/2 \end{cases} \tag{3}$$

where

$$Q(z) \equiv \int_z^\infty f(t)\ dt$$

is the well known Q function, which is tabulated in many references or can easily be computed using simple algorithms.

The SER, P_s, is given by:

$$P_s = \sum_{m=1}^{M-1} A_m \tag{4}$$

Note that for any k, SER can be represented as the sum of probabilities for two half-planes minus that for the overlapped area $R_{M/2}$:

$$P_s = \sum_{m=1}^{M/2} A_m + \sum_{m=M/2}^{M-1} A_m - A_{M/2} = 2 \cdot Q\big[d_k \sin\big(a_k\big)\big] - A_{M/2}$$

For $k = 1$, since $A_{M/2} = A_1$, $P_s = Q\big[d_1 \sin\big(a_1\big)\big]$. For $k = 2$, since $R_{M/2}$ is a quadrant, we have $A_{M/2} = Q^2\big[d_2 \sin\big(a_2\big)\big]$. For $k \geqslant 3$, the exact $A_{M/2}$ must be evaluated by Eq. (3) or similar expressions available in the literature. Also many bounds have been developed for $A_{M/2}$. See Ref. 1 for an example.

III. Bit Error Rate

Let w_m be the Hamming weight of the bits assigned to symbol m (see Table 1). Since the probability of bit error is w_m/k when $X = 0$ and $y \in R_m$, we have the following expression for average BER, P_b:

$$P_b = \frac{1}{k} \cdot \sum_{m=1}^{M-1} w_m A_m \tag{5}$$

Since $w_m \geqslant 1$ for all $m \neq 0$, from Eqs. (5) and (4),

$$P_b \geqslant \frac{1}{k} \cdot \sum_{m=1}^{M-1} A_m = \frac{1}{k} \cdot P_s \tag{6}$$

This "conventional" lower bound appears frequently in the literature (e.g., Ref. 2, p. 231) and is known to be very tight for large E_b/N_0, since, in such a case, most errors occur in adjacent regions. For $k = 1$ $(M = 2)$, $P_b = P_s = Q\big[d_1 \sin\left(a_1\right)\big]$. For $k = 2$ $(M = 4)$, since $A_3 = A_1$:

$$P_b = \big\{A_1 + 2A_2 + A_3\big\}/2 = A_1 + A_2 = Q\big[d_2 \sin\big(a_2\big)\big] \tag{7}$$

Since $d_2 \sin\left(a_2\right) = d_1 \sin\left(a_1\right)$, BER for BPSK and QPSK are identical. This fact also has been noticed in the literature (e.g., Ref. 2, p. 234).

However, for larger k, there has been no effort toward finding the exact BER expression or its tight bounds. From Eq. (1), we have the following:

for $k = 3$ $(M = 8)$,

$$P_b = \left\{A_1 + 2A_2 + A_3 + 2A_4 + 3A_5 + 2A_6 + A_7\right\}/3$$

$$= (2/3) \cdot \left\{A_1 + A_2 + A_3 + A_4 + A_5 + A_6\right\}$$

$$= (2/3) \cdot \Big\{Q\left[d_3 \sin\left(a_3\right)\right]$$

$$+ Q\left[d_3 \sin\left(3a_3\right)\right] \cdot \left(1 - Q\left[d_3 \sin\left(a_3\right)\right]\right)\Big\} \tag{8}$$

Hence, we have a closed-form expression for the exact BER of 8-ary PSK only in terms of the Q functions!

For $k = 4$ ($M = 16$),

$$P_b = (2/4) \cdot \left\{\sum_{m=1}^{8} A_m + \sum_{m=2}^{5} A_m + A_5 + 2A_6 + A_7\right\} \tag{9}$$

The probabilities for the half-plane and for the quadrant can be easily found in terms of the Q function. The exact values of the remaining terms can be found by Eq. (2) or Eq. (3). Or, instead, one may use the following upper and lower bounds:

$$\sum_{m=5}^{8} A_m < A_5 + 2A_6 + A_7 < \sum_{m=4}^{7} A_m \tag{10}$$

For $k = 5$ ($M = 32$),

$$P_b = (2/5) \cdot \Big\{\sum_{m=1}^{16} A_m + \sum_{m=2}^{9} A_m + \sum_{m=5}^{12} A_m - \sum_{m=7}^{14} A_m$$

$$+ \sum_{m=9}^{16} A_m + 2A_{10} + 2A_{11} + A_{12}$$

$$+ 2A_{13} + 2A_{14} - A_{16}\Big\} \tag{11}$$

These summations can be found in terms of Q functions, while for the remaining terms one may use the following bounds:

$$\sum_{m=10}^{17} A_m < 2A_{10} + 2A_{11} + A_{12} + 2A_{13} + 2A_{14} - A_{16}$$

$$< \sum_{m=8}^{15} A_m \tag{12}$$

For M-ary PSK with larger M, we can also find very tight upper and lower bounds on BER. These will not be described here, except for the following simple lower bound on BER, which is much tighter than the conventional lower bound of Eq. (6): for $k \geqslant 4$,

$$P_b < (2/k) \cdot \left\{Q\left[d_k \sin\left(a_k\right)\right] + Q\left[d_k \sin\left(3a_k\right)\right]\right\} \tag{13}$$

The exact BER of M-ary PSK is shown in Table 2 for $M = 2$ to 64. Also the accuracy of the bounds on BER is tested in terms of the relative error (difference between the exact BER value and its bound, normalized to the exact value). Those relative errors are tabulated in Table 2, where we see that our bounds are extremely tight for all values of E_b/N_0 of interest.

IV. Conclusions and Discussion

For coherent M-ary PSK over an AWGN channel with Gray code bit mapping we can draw the following conclusions: (1) a general expression for BER is found; (2) a closed-form expression for the exact BER of 8-ary PSK is derived; (3) a lower bound on BER is found which is tighter than the conventional lower bound; (4) in particular, for 16- and 32-ary PSK, much tighter upper and lower bounds on BER are obtained.

The technique developed in this paper can also be used for finding the BER of M-ary PSK signals with either coherent or differentially coherent detection, as long as the noise is circularly symmetric. This was pointed out by Dr. Marvin K. Simon (Ref. 3) after reading the original manuscript of this paper. Using Eqs. (9) and (10) of Ref. 4, he derived the following simple integral expression for the exact BER of 8-ary DPSK over an AWGN channel.

$$P_b = (2/3)\left\{F\left(13a_3\right) - F\left(a_3\right)\right\}$$

where

$$F(\phi) = \frac{-\sin\phi}{4\pi} \int_{-\pi/2}^{\pi/2} \frac{e^{-d_3^2/2 \cdot (1 - \cos\phi \cdot \cos\theta)}}{1 - \cos\phi \cdot \cos\theta}\, d\theta$$

References

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3. Private communication from M. K. Simon of JPL, January, 1985.

4. Pawula, R. F., Rice, S. O., and Roberts, J. H., "Distribution of the Phase Angle Between Two Vectors Perturbed by Gaussian Noise," *IEEE Trans. Commun.*, Vol. COM-30, pp. 1828-1841, August 1982.

Table 1. Gray bit mapping and Hamming weight *W*

m	bits	W_m	m	bits	W_m
$k = 2$ $(M = 4)$			$k = 5$ $(M = 32)$		
0	0 0	0	0	0 0 0 0 0	0
1	0 1	1	1	0 0 0 0 1	1
2	1 1	2	2	0 0 0 1 1	2
3	1 0	1	3	0 0 0 1 0	1
			4	0 0 1 1 0	2
$k = 3$ $(M = 8)$			5	0 0 1 1 1	3
0	0 0 0	0	6	0 0 1 0 1	2
1	0 0 1	1	7	0 0 1 0 0	1
2	0 1 1	2	8	0 1 1 0 0	2
3	0 1 0	1	9	0 1 1 0 1	3
4	1 1 0	2	10	0 1 1 1 1	4
5	1 1 1	3	11	0 1 1 1 0	3
6	1 0 1	2	12	0 1 0 1 0	2
7	1 0 0	1	13	0 1 0 1 1	3
			14	0 1 0 0 1	2
$k = 4$ $(M = 16)$			15	0 1 0 0 0	1
0	0 0 0 0	0	16	1 1 0 0 0	2
1	0 0 0 1	1	17	1 1 0 0 1	3
2	0 0 1 1	2	18	1 1 0 1 1	4
3	0 0 1 0	1	19	1 1 0 1 0	3
4	0 1 1 0	2	20	1 1 1 1 0	4
5	0 1 1 1	3	21	1 1 1 1 1	5
6	0 1 0 1	2	22	1 1 1 0 1	4
7	0 1 0 0	1	23	1 1 1 0 0	3
8	1 1 0 0	2	24	1 0 1 0 0	2
9	1 1 0 1	3	25	1 0 1 0 1	3
10	1 1 1 1	4	26	1 0 1 1 1	4
11	1 1 1 0	3	27	1 0 1 1 0	3
12	1 0 1 0	2	28	1 0 0 1 0	2
13	1 0 1 1	3	29	1 0 0 1 1	3
14	1 0 0 1	2	30	1 0 0 0 1	2
15	1 0 0 0	1	31	1 0 0 0 0	1

Table 2. Exact BER values of *M*-ary PSK with Gray mapping

E_b/N_0, dB	P_b (M = 2,4)	P_b (M = 8)	P_b (M = 16)	P_b (M = 32)	P_b (M = 64)
-5.0	2.132E-01	2.468E-01	2.867E-01	3.185E-01	3.376E-01
-4.0	1.861E-01	2.217E-01	2.646E-01	3.003E-01	3.242E-01
-3.0	1.584E-01	1.961E-01	2.420E-01	2.817E-01	3.099E-01
-2.0	1.306E-01	1.708E-01	2.191E-01	2.629E-01	2.951E-01
-1.0	1.038E-01	1.461E-01	1.965E-01	2.442E-01	2.800E-01
0.0	7.865E-02	1.227E-01	1.745E-01	2.256E-01	2.649E-01
1.0	5.628E-02	1.008E-01	1.535E-01	2.073E-01	2.497E-01
2.0	3.751E-02	8.061E-02	1.338E-01	1.892E-01	2.345E-01
3.0	2.288E-02	6.225E-02	1.155E-01	1.714E-01	2.194E-01
4.0	1.250E-02	4.589E-02	9.865E-02	1.538E-01	2.043E-01
5.0	5.954E-03	3.186E-02	8.292E-02	1.368E-01	1.895E-01
6.0	2.388E-03	2.048E-02	6.816E-02	1.207E-01	1.747E-01
7.0	7.727E-04	1.195E-02	5.429E-02	1.055E-01	1.599E-01
8.0	1.909E-04	6.181E-03	4.145E-02	9.147E-02	1.452E-01
9.0	3.363E-05	2.748E-03	2.998E-02	7.840E-02	1.307E-01
10.0	3.872E-06	1.011E-03	2.025E-02	6.614E-02	1.165E-01
11.0	2.613E-07	2.937E-04	1.256E-02	5.451E-02	1.030E-01
12.0	9.006E-09	6.338E-05	7.010E-03	4.349E-02	9.027E-02
13.0		9.417E-06	3.427E-03	3.325E-02	7.848E-02
14.0		8.756E-07	1.421E-03	2.406E-02	6.752E-02
15.0		4.516E-08	4.789E-04	1.627E-02	5.724E-02
16.0			1.246E-04	1.010E-02	4.747E-02
17.0			2.342E-05	5.642E-03	3.819E-02
18.0			2.925E-06	2.763E-03	2.950E-02
19.0			2.187E-07	1.147E-03	2.163E-02
20.0			8.573E-09	3.876E-04	1.486E-02
21.0				1.011E-04	9.417E-03
22.0				1.907E-05	5.394E-03
23.0				2.393E-06	2.725E-03
24.0				1.799E-07	1.177E-03
25.0				7.099E-09	4.176E-04
26.0					1.159E-04
27.0					2.361E-05
28.0					3.264E-06

Table 3. Accuracy of the upper and lower bounds on BER

	Relative error of the bound on BER found from:				
E_b/N_0, dB	(6), Lower	(13), Lower	(10), Lower	(10), Upper	: (M = 16)
-3.0	2.82E-01	6.16E-03	1.53E-03	1.40E-02	
0.0	1.67E-01	9.66E-04	2.15E-04	3.11E-03	
3.0	5.72E-02	2.05E-05	4.06E-06	1.05E-04	
6.0	6.30E-03	8.44E-09	1.48E-09	8.17E-08	
9.0	7.90E-05	1.58E-15	2.48E-16	3.90E-14	
E_b/N_0, dB	(6), Lower	(13), Lower	(12), Lower	(12), Upper	: (M = 32)
-3.0	4.13E-01	4.70E-02	1.65E-03	3.25E-03	
0.0	3.29E-01	1.15E-02	1.32E-04	3.83E-04	
3.0	2.28E-01	6.30E-04	8.37E-07	3.84E-06	
6.0	1.11E-01	1.78E-06	3.68E-11	2.89E-10	
9.0	2.47E-02	1.43E-11	8.28E-20	1.25E-18	

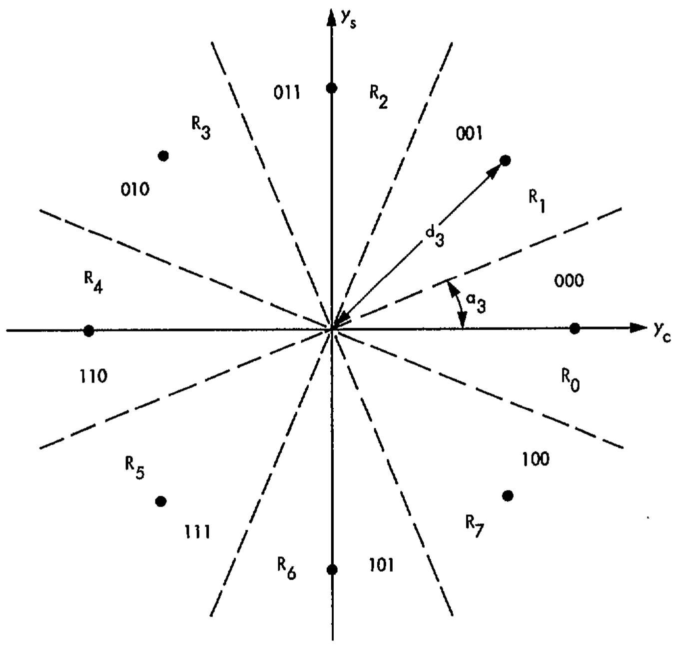

Fig. 1. 8-ary PSK signal space with optimum decision regions